Time-Inconsistent Mean-Field Stochastic LQ Problem: Open-Loop Time-Consistent Control

Yuan-Hua Ni Ji-Feng Zhang Miroslav Krstic

Abstract—This paper is concerned with the open-loop time-consistent solution of time-inconsistent mean-field stochastic linear-quadratic optimal control. Different from standard stochastic linear-quadratic problems, both the system matrices and the weighting matrices are depending on the initial times, and the conditional expectations of the control and state enter quadratically into the cost functional. Such features will ruin Bellman’s principle of optimality and result in the time-inconsistency of optimal control. Based on the dynamical nature of the systems involved, a kind of open-loop time-consistent equilibrium control is investigated in this paper. It is shown that the existence of open-loop equilibrium control for a fixed initial pair is equivalent to the solvability of a set of forward-backward stochastic difference equations with stationary condition and convexity condition. By decoupling the forward-backward stochastic differential equations, necessary and sufficient conditions in terms of linear difference equations and generalized difference Riccati equations are given for the existence of open-loop equilibrium control for a fixed initial pair. Moreover, the existence of open-loop time-consistent equilibrium controls for all the initial pairs is shown to be equivalent to the solvability of a set of coupled constrained generalized difference Riccati equations and two sets of constrained linear difference equations.

Index Terms—Mean-field theory, stochastic linear-quadratic optimal control, time-inconsistency, forward-backward stochastic difference equation

I. INTRODUCTION

A. Time-consistency vs. time-inconsistency

Though not mentioned frequently, time-consistency is indeed an essential notion in optimal control theory, which relates to Bellman’s principle of optimality. To see this, recall a standard discrete-time stochastic optimal control problem, whose system dynamics and cost functional are given, respectively, by

\[
\begin{align*}
X_{k+1} &= f(k, X_k, u_k, w_k), \\
X_t &= x \in \mathbb{R}^n, \quad k \in \mathbb{T}_t, \quad t \in \mathbb{T},
\end{align*}
\]

and

\[
J(t, x; u) = \sum_{k=t}^{N-1} \mathbb{E}[e^{-\delta(k-t)} L(k, X_k, u_k)] + \mathbb{E}[e^{-\delta(N-t)} h(X_N)].
\]

Here, \( T_t = \{t, \cdots, N-1\}, T = \{0, 1, \cdots, N-1\}, \) and \( N \) is a positive integer; \( \{X_k, k \in T_t\} \) and \( \{u_k, k \in T_t\} \) are the state process and the control process, respectively; \( \{w_k, k \in T\} \) is a stochastic disturbance process; \( \mathbb{E} \) is the operator of mathematical expectation. Without loss of generality, the functions \( f, L \) and \( h \) are assumed bounded. Let \( \mathcal{U}[t, N-1] \) be a set of admissible controls. Then, we have the following optimal control problem.

**Problem (C).** Letting \((t, x) \in T \times \mathbb{R}^n\), find a \( \bar{u} \in \mathcal{U}[t, N-1] \) such that

\[
J(t, x; \bar{u}) = \inf_{u \in \mathcal{U}[t, N-1]} J(t, x; u).
\]

Above problem will be called Problem (C) for the initial pair \((t, x)\), and Problem (C) for other initial pairs can be similarly formulated. Any \( \bar{u} \in \mathcal{U}[t, N-1] \) satisfying (3) is called an optimal control for the initial pair \((t, x)\), and \( \bar{X} = \{X_k = X(k; t, x, \bar{u}), k \in T_t\} \) is the corresponding optimal trajectory. Furthermore, \((\bar{X}, \bar{u})\) is referred to as an optimal pair for the initial pair \((t, x)\).

Let \((\bar{X}, \bar{u})\) be an optimal pair for the initial pair \((t, x)\); as the dynamics evolves, we indeed face a family of optimal control problems, namely, Problem (C) for the initial pairs \(\{(k, X_k), k \in T_t\}\). Bellman’s principle of optimality tells us that the optimal controls of this family of problems are interrelated, namely, for any \(\tau \in T_{t+1} = \{t+1, \cdots, N-1\}, \bar{u|\tau} = \{\bar{u}_\tau, \cdots, \bar{u}_{N-1}\}\) (the restriction of \(\bar{u}\) on \(T_\tau = \{\tau, \cdots, N-1\}\) is an optimal control of Problem (C) for the initial pair \((\tau, \bar{X}_\tau)\). This property is the cornerstone of Bellman’s dynamic programming and is referred to as the time-consistency of optimal control, which is essential to handle optimal control problems like Problem (C) and its continuous-time counterpart. In such situation, we call that Problem (C) is time-consistent.

However, the time-consistency fails quite often in many situations. For instance, when the exponential discounting function \(e^{-\delta(k-t)}\) in (2) is replaced by other discounting functions, the corresponding problem is not time-consistent, i.e., time-inconsistent; see examples in [5], [19] about the hyperbolic discounting and quasi-geometric discounting. In addition, when the conditional expectations of the state and/or control enters nonlinearly into the cost functional, the considered optimal control problems are time-inconsistent too;
a notable example is the mean-variance utility [2], [5], [9], [11], [22], [24]. In such case, the smoothing property of conditional expectation will not be sufficient to ensure the time-consistency of optimal control.

B. Literature review

Problems with nonlinear terms of conditional expectation (in the cost functional) are classified into the mean-field stochastic optimal control [37]. In [22], recognizing the time-inconsistency (called nonseparability there), Li and Ng derived the optimal policy of multi-period mean-variance portfolio selection by using an embedding scheme. Note that the optimal policy of [22] is with respect to the initial pair, i.e., it makes sense to be optimal only when viewed at the initial time. This derivation is called the pre-commitment optimal solution now.

Pre-commitment optimal solution is a static notion, which maps the considered initial pair into an admissible control set. By applying a pre-commitment optimal control (for an initial pair), its restriction to the tail time horizon is not an optimal control for the inter temporal initial pair. This static trait conflicts with the dynamic nature of (time-inconsistent) optimal control, as the time is involved in the problem setting. Though the static solution is of some practical and theoretical values, it neglects and has not really addressed the time-inconsistency. Differently, another approach handles the time-inconsistency in a dynamic manner; instead of seeking a pre-commitment optimal control, some kinds of equilibrium solutions are dealt with. This is mainly motivated by practical applications in economics and finance, and has recently attracted considerable interest and efforts.

The explicit formulation of time-inconsistency was initiated by Strotz [29] in 1955, whereas its qualitative analysis can be traced back to the work of Smith [28]. Strotz studied the general discounting problem, and in the discrete-time case, his idea is to tackle the time-inconsistency by a lead-follower game with hierarchical structure. Specifically, controls at different time points were viewed as different selves (players), and every self integrated the policies of his successor into his own decision. By a backward procedure, the equilibrium policy (if it exists) was obtained. Inspired by Strotz and intending to tackling practical problems in economics and finance, hundreds of works were concerned with time-inconsistency of dynamic systems described by ordinary difference or differential equations; see, for example, [12], [13], [15], [19], [20], [26] and references therein. Unfortunately, as pointed out by Ekeland [12], [13], it is hard to prove the existence of Strotz's equilibrium policy. Therefore, it is necessary and of great importance to develop a general theory on time-inconsistent optimal control. This, on the one hand, can enrich the optimal control theory, and on the other hand, can provide instructive methodology to push the solvability of practical problems. Recently, this topic has attracted considerable attention from the theoretic control community; see, for example, [5], [17], [18], [30], [32], [34], [37] and references therein.

For the time-inconsistent LQ problems, two kinds of time-consistent equilibrium solutions are studied, which are the open-loop equilibrium control and the closed-loop equilibrium strategy [17], [18], [32], [34], [37]. The separate investigations of such two formulations are due to the fact that in the dynamic game theory, open-loop control distinguishes significantly from closed-loop strategy [3], [36]. To compare, open-loop formulation is to find an open-loop equilibrium “control”, while the “strategy” is the object of closed-loop formulation. By a strategy, we mean a decision rule that a controller uses to select a control action based on the available information set. Mathematically, a strategy is a mapping or operator on the information set. When substituting the available information into a strategy, the open-loop value or open-loop realization of this strategy is obtained. Strotz’s equilibrium solution [29] is essentially a closed-loop equilibrium strategy, which is further elaborately developed by Yong to the LQ optimal control [32], [37] as well as the nonlinear optimal control [34], [33]. In contrast, open-loop equilibrium control is extensively studied by Hu-Jin-Zhou [17], [18] and Yong [37]. In particular, the closed-loop formulation can be viewed as the extension of Bellman’s dynamic programming, and the corresponding equilibrium strategy (if it exists) is derived by a backward procedure [32], [33], [34], [37]. Differently, the open-loop equilibrium control is characterized via the maximum-principle-like methodology [17], [18].

Portfolio selection is to seek a best allocation of wealth among a basket of securities. The (single-period) mean-variance formulation is pioneered by Markowitz [24] in 1952, which is the cornerstone of modern portfolio theory and is widely used in both academia and industry. The multi-period mean-variance portfolio selection is the natural extension of [24], which has been extensively studied. Until 2000 and for the first time, Li-Ng [22] and Zhou-Li [38] reported the analytical pre-commitment optimal policies for the discrete-time case and the continuous-time case, respectively. Noted above, multi-period mean-variance portfolio selection is a particular example of time-inconsistent optimal control; the recent developments in time-inconsistent optimal control and the revisits of multi-period mean-variance portfolio selection [2], [6], [9], [10], [17], [18] are mutually stimulated.

It is noted that some nondegenerate assumptions are posed in [2], [6], [9], [10], [17], [18]. Specifically, the volatilities of the stocks in [2], [6], [17], [18] and the return rates of the risky securities in [9], [10] are assumed to be nondegenerate. To make the formulation more practical, it is natural to consider, at least in theory, how to generalize these results to the case where degeneracy is allowed. In fact, mean-variance portfolio selection problems with degenerate covariance matrices may date back to 1970s. In [7] or the “corrected” version [27], Buser et al propose the single-period version with possibly singular covariance matrix. Clearly, such class of problems are more general than the classical ones [24], and more consistent with the reality.

To address the case with possible degenerate return rates, it is better to put multi-period mean-variance portfolio selection within the framework of time-inconsistent mean-field stochastic LQ optimal control (with indefinite weighting matrices), which has not been established yet. Note that the running weighting matrices in [17], [18], [32], [34], [37] are assumed to be nonnegative definite and positive definite.
For standard time-consistent indefinite stochastic LQ optimal control, readers are referred to, for example, [1], [8], [31] and reference therein.

C. Contents of this paper

In this paper, we shall investigate a time-inconsistent indefinite mean-field stochastic LQ optimal control problem. The matrices in system dynamics and cost functional are also dependent on the initial times; this is an extension of the general discounting functions that are in cost functionals. The contents of this paper are as follows.

The notion of open-loop equilibrium control is introduced in Section II, which is a discrete-time counterpart of that for the continuous-time problem [17], [18]. Different from the pre-commitment optimal control, the equilibrium control is only locally optimal in an infinitesimal sense. Furthermore, the open-loop equilibrium control is defined for a fixed initial time-state pair; its existence is shown to be equivalent to some stationary condition and convexity condition, which are involved with a set of forward-backward stochastic difference equations (FBSΔEs, for short). Furthermore, necessary and sufficient conditions are obtained, respectively, for the stationary condition and the convexity condition; and by combining them, the existence of open-loop equilibrium control is further characterized.

The convexity condition is equivalent to the nonnegative definiteness of some matrices relating to a set of linear difference equations (LDEs, for short), which is called the solvability of those constrained LDEs. The stationary condition is characterized via a property about the ranges of some matrices that are involved with another set of LDEs and a set of generalized difference Riccati equations (GDREs, for short). If we further let the initial pair vary, some neater result about the existence of open-loop equilibrium control will be obtained. Specifically, for any initial pair Problem (LQ) admitting an open-loop equilibrium control is shown to be equivalent to that two sets of constrained LDEs (39) (41) and a set of constrained GDREs (40) are solvable. It is worth pointing out that (if it is solvable) the set of GDREs (40) does not have symmetric structure, i.e., its solution is not symmetric. Furthermore, all the open-loop equilibrium controls are obtained.

As application of the derived theory, Section V investigates the multi-period mean-variance portfolio selection. Necessary and sufficient condition is given on the existence of open-loop equilibrium portfolio control, which is completely characterized by the returns of the risky and riskless assets. If the return rates of the risky securities are nondegenerate, the equilibrium portfolio control will exist.

From our derived results, we have the following remarks.

- Most existing results about time-inconsistent LQ problems are for the continuous-time case [17], [18], [32], [34], [37], and the study of discrete-time case is lagging behind. Noted above, the discrete-time multi-period mean-variance portfolio selection is a notable example of discrete-time time-inconsistent LQ problems, and its full investigation motivates and needs to develop general theory about discrete-time time-inconsistent LQ optimal control. This is the aim of the paper.

- The novelties of this paper are as follows. Firstly, no definiteness constraint is posed on the weighting matrices of cost functional, namely, the considered problem is an indefinite LQ optimal control. On the one hand, the indefinite setting provides a maximal capacity to model and deal with LQ-type problems, whose study will generalize existing results to some extent. On the other hand and most importantly, general explicit answers have not been reported about whether or not the definite weighting matrices could ensure the existence of open-loop equilibrium control for a time-inconsistent LQ problem. Therefore, the essential and weakest conditions are much desired for ensuring the existence of open-loop equilibrium control; and it is not necessary to pose the definiteness constraint on the weighting matrices. Secondly, necessary and sufficient conditions are obtained on the existence of open-loop equilibrium control of Problem (LQ) for both the case with a fixed initial pair and the case with all the initial pairs. The conditions are in terms of discrete-time LDEs and GDREs, which are easy to be verified by iteratively solving the LDEs and GDREs.

Thirdly, necessary and sufficient condition is derived on the existence of open-loop equilibrium portfolio control of multi-period mean-variance portfolio selection (Problem (MV)). The obtained condition is completely characterized by the returns of the risky and riskless assets. If the return rates of the risky securities are nondegenerate (this is the common assumption in the literature), the equilibrium portfolio control will exist.

In [23], a simplified version of Problem (LQ) is considered, where there are no mean-field terms in the system dynamics and cost functional. Hence, this paper is a continuation of [23]. Concerned with the necessary and sufficient condition on the existence of open-loop equilibrium pair, [23] just gives a very simplified version of Theorem 4.1 of this paper. This is because in [23] we do not have a result similar to Lemma 3.3, which gives the representation of the FBSΔE’s backward state via the forward state. Furthermore, the case with a fixed initial pair is not thoroughly investigated, while necessary and sufficient condition in terms of LDEs and GDREs is given in Theorem 3.4. If the system dynamics and cost functional are both independent of the initial time, the corresponding LQ problem will be a dynamic version of that considered in [25], where the conditional expectation operators are replaced by the expectation operators. For more details on mean-field stochastic optimal control and related mean-field games, we refer to [4], [11], [14], [16], [21], [25], [35] and the references therein.

The rest of this paper is organized as follows. Section II introduces the notion of open-loop equilibrium control of Problem (LQ). In Section III and Section IV, necessary and sufficient conditions on the existence of open-loop equilibrium control are presented for both the case with a fixed initial pair and the case with all the initial pairs. Section V studies the multi-period mean-variance portfolio selection, and some concluding remarks are given in Section VI.
II. OPEN-LOOP EQUILIBRIUM CONTROL

Consider the following controlled stochastic difference equation (SDE, for short)

\[
X_{k+1}^t = (A_{t,k}X_k^t + \bar{A}_{t,k}E_tX_k^t + B_{t,k}u_k + f_{t,k} + B_t\eta_k) \quad \text{for } t \in \mathbb{T}_k,
\]

where \(A_{t,k}, \bar{A}_{t,k}, C_{t,k}^i, C_{t,k}^i \in \mathbb{R}^{n \times n}\), \(B_{t,k}, B_t \in \mathbb{R}^{n \times n}\) and \(f_{t,k}, \eta_k \in \mathbb{R}^n\) are deterministic matrices. In (4), the noise process \(w_k = (w_k^1, \ldots, w_k^p)^T, k \in \mathbb{T}\) is assumed to be a vector-valued martingale difference sequence defined on a probability space \((\Omega, \mathcal{F}, P)\) with

\[
E_k[w_k] = 0, \quad E_k[w_kw_{k+1}^T] = \Gamma_k, \quad k \geq 0.
\]

Thus, \(E_t\) in (4) is the conditional mathematical expectation \(E[\cdot | \mathcal{F}_t]\), where \(\mathcal{F}_t = \sigma\{w_l, l = 0, 1, \ldots, t - 1\}\), and \(\mathcal{F}_0\) is understood as \(\{\Omega\}\). Furthermore, \(\Gamma_k = (\gamma_{k,j})_{p \times p}\) is assumed to be symmetric.

The cost functional associated with the system (4) is

\[
J(t, x; u) = \sum_{k=t}^{T} \left( \left( E_t X_k^t \right)^T Q_{t,k} X_k^t + \left( E_t u_k \right)^T R_{t,k} E_t u_k + 2 \rho_{t,k}^T X_k^t \right) + 2 \rho_{t,k}^T G_{t,k} E_t X_k^t + \left( E_t X_k^t \right)^T G_{t,k} E_t X_k^t + 2 \epsilon_t^T E_t X_k^t,
\]

which is called the equilibrium state corresponding to \(u^{t,x}\).

Noting that \(u^{t,x}|_{T_k} = (u_k^{t,x}, \ldots, u_{T_k}^{t,x})\) on the right-hand side of (8) differs from \(u^{t,x}|_{T_k}\), only at time instant \(k\). Intuitively, the cost functional will increase if one deviates from \(u^{t,x}\). Hence, \(u_k^{t,x}, \ldots, u_{T_k}^{t,x}\) can be viewed as an equilibrium of a multi-person game with hierarchical structure. By its definition, \(u^{t,x}\) is time-consistent in the sense that for any \(k \in \mathbb{T}_t, u^{t,x}|_{T_k}\) is an open-loop equilibrium control for the initial pair \((k, X_k^{t,x})\).

III. PROBLEM (LQ) FOR A FIXED INITIAL PAIR

A. The first characterization on the existence of open-loop equilibrium control

Throughout Section III, we will study Problem (LQ) for the fixed initial pair \((t, x)\), which will be simply denoted as Problem (LQ)\(_{t,x}\). Firstly, a difference formula of cost functionals is given.

**Lemma 3.1:** Let \(\zeta \in L^2_T(k, \mathbb{R}^n), u = \{u_k, k \in \mathbb{T}_k\} \in L^2(T_k, \mathbb{R}^m), \bar{u}_k \in L^2_T(k, \mathbb{R}^m)\) and \(\lambda \in \mathbb{R}\). Then, we have

\[
J(k, \zeta; (u_k + \lambda u_k, u_{\lambda+1}^{t,x} - \lambda)) - J(k, \zeta; u) = \lambda^2 \dot{J}(k, 0; \bar{u}_k) + 2\lambda \left( R_{t,k}u_k + B_{t,k}^T E_t Z_{k+1}^T \right) + \sum_{i=1}^{p} \left( D_{t,k}^i u_k^{t,x} + d_{k,i}^T \right) \Theta_{k+1}^T \bar{u}_k,
\]

where \(u|_{x_{k+1}} = \{u_{k+1}^{t,x}, \ldots, u_{N-1}\}\) and

\[
\dot{J}(k, 0; \bar{u}_k) = E_k[\bar{u}_k^T R_{t,k} u_k] + \sum_{k=0}^{N-1} E_k \left[ (Y^k_{\lambda, \bar{u}_k})^T Q_k u_k + \left( E_k Y^k_{\lambda, \bar{u}_k} \right)^T Q_{k,t} E_k Y^k_{\lambda, \bar{u}_k} \right] + E_k \left[ (Y^k_{\lambda, \bar{u}_k})^T G_k Y^k_{\lambda, \bar{u}_k} \right] + E_k \left[ (Y^k_{\lambda, \bar{u}_k})^T G_k Y^k_{\lambda, \bar{u}_k} \right].
\]
with
\[
\begin{aligned}
Y_{k+1}^{k,u} & = A_k R Y_{k+1}^{k,u} + A_k E_k E_k X_{k}^{k,u} \\
& + \sum_{i=1}^{p} (C_{k,i} X_{k}^{k,u} + C_{k,i} E_k X_{k}^{k,u}) w_i \\
Y_{k}^{k,u} & = B_{k,u} u_k + \sum_{i=1}^{p} D_{k,u} w_i,
\end{aligned}
\] (14)

Furthermore, $Z_{k+1}^{k}$ in (12) is computed via the following FBSDE
\[
\begin{aligned}
X_{k+1}^{k} & = (A_k R X_{k}^{k} + A_k E_k E_k X_{k}^{k} \\
& + B_{k,u} u_k + B_{k} E_k E_k u_k + f_k, t) \\
& + \sum_{i=1}^{p} (C_{k,i} X_{k}^{k} + C_{k,i} E_k X_{k}^{k}) w_i \\
& + D_{k,u} u_k + D_{k} E_k E_k u_k + d_k, t),
\end{aligned}
\]
\[
\begin{aligned}
Z_{k+1}^{k} & = Q_{k, u} X_{k+1}^{k} + Q_{k} E_k E_k X_{k+1}^{k} \\
& + A_{k}^{T} E_k E_k Z_{k+1}^{k} + A_{k}^{T} E_k E_k Z_{k+1}^{k} \\
& + \sum_{i=1}^{p} [(C_{k,i})^{T} E_k (Z_{k+1}^{k}, w_i)] \\
& + (C_{k})^{T} E_k (Z_{k+1}^{k}, w_i),
\end{aligned}
\]
\[
\begin{aligned}
X_{k}^{k} & = \zeta, \\
Z_{k}^{k} & = G_k X_{k}^{k} + G_k E_k X_{k}^{k} + g_k,
\end{aligned}
\]
$\ell \in T_k.$

Proof. See Appendix A.

From Lemma 3.1, we have the following result.

Theorem 3.1: The following statements are equivalent.

i) Problem (LQ)$_k$ admits an open-loop equilibrium control.

ii) The following assertions hold.

a) The convexity condition
\[
\inf_{\bar{u}_k \in \Omega_{k}^{\mu}(T_k)} J(k, 0; \bar{u}_k) \geq 0, \quad k \in T_k
\] (16)
is satisfied, where $\bar{J}(k, 0; \bar{u}_k)$ is given in (13).

b) There exists a $u^{t,x,*} \in L_{k}^{2}(T_k; \mathbb{R}^{m})$ such that the stationary condition
\[
0 = R_{k,V} u_{t,x,*} + B_{k,T} E_k Z_{t,x}^{k,*}
+ \sum_{i=1}^{p} (D_{k,B}^{i} E_k (Z_{t,x}^{k,*}, w_i)) + \rho_{k,k}, \quad k \in T_k
\] (17)
is satisfied. Here, $Z_{t,x}^{k,*}$ is computed via the FBSDE
\[
\begin{aligned}
X_{t+1}^{k,x} & = (A_k R X_{t}^{k,x} + A_k E_k E_k X_{t}^{k,x} \\
& + B_{k,u} u_{t,x,*} + B_{k} E_k E_k u_{t,x,*} + f_k, t) \\
& + \sum_{i=1}^{p} (C_{k,i} X_{t}^{k,x} + C_{k,i} E_k X_{t}^{k,x}) w_i \\
& + D_{k,u} u_{t,x,*} + D_{k} E_k E_k u_{t,x,*} \\
& + d_k, t),
\end{aligned}
\]
\[
\begin{aligned}
Z_{t}^{k,x} & = Q_{k, u} X_{t}^{k,x} + Q_{k} E_k E_k X_{t}^{k,x} \\
& + A_{k}^{T} E_k E_k Z_{t}^{k,x} + A_{k}^{T} E_k E_k Z_{t}^{k,x} \\
& + \sum_{i=1}^{p} [(C_{k,i})^{T} E_k (Z_{t}^{k,x}, w_i)] \\
& + (C_{k})^{T} E_k (Z_{t}^{k,x}, w_i),
\end{aligned}
\]
\[
\begin{aligned}
X_{k}^{k,x} & = X_{t,x}^{k}, \\
Z_{k}^{k,x} & = G_k X_{k}^{k,x} + G_k E_k X_{k}^{k,x} + g_k,
\end{aligned}
\]
$\ell \in T_k.$

and the initial state $X_{k}^{t,x,*}$ of the forward SDE of (18) is computed via
\[
\begin{aligned}
X_{t,x}^{t,x,*} & = [A_k R X_{t,x}^{t,x,*} + B_{k,u} u_{t,x,*} + f_k, t] \\
& + \sum_{i=1}^{p} [C_{k,i} X_{t,x}^{t,x,*} + D_{k,u} u_{t,x,*} \\
& + d_k, t)] w_i.
\end{aligned}
\]

Under any of the above conditions, $u^{t,x,*}$ given in ii) is an open-loop equilibrium control for the initial pair $(t, x)$.

Proof. See Appendix B.

Remark 3.1: As the stationary condition (17) holds for $k \in T_k$, we have a set of FBSDEs, which are coupled with (19) via the initial states $X_{k}^{t,x,*} = X_{k}^{k,x,*}, k \in T_k$.

B. Convexity condition

This subsection studies the convexity condition (16). Firstly, we give a compact form of $\hat{J}(k, 0; \bar{u}_k)$.

Lemma 3.2: $J(k, 0; \bar{u}_k)$ can be expressed as
\[
\hat{J}(k, 0; \bar{u}_k) = \bar{u}_{T}^{T} \bar{W}_{k} \bar{u}_{k}
\]
(20)
where
\[
\bar{W}_{k} = \bar{R}_{k,k} + \bar{B}_{k,k}^{T} \bar{P}_{k,k+1} \bar{B}_{k,k}
+ \sum_{i,j=1}^{p} \gamma_{ij}^{k} (\bar{D}_{k,k}^{i})^{T} \bar{P}_{k,k+1} \bar{D}_{k,k}^{i}
\]
(21)

with $\bar{P}_{k,k+1}$ and $\bar{P}_{k,k+1}$ computed via
\[
\begin{aligned}
P_{k,t} & = Q_{k,t} + A_{k,t}^{T} P_{k,t+1} A_{k,t} \\
& + \sum_{i,j=1}^{p} \gamma_{ij}^{k} (C_{k,t})^{T} P_{k,t+1} C_{k,t},
\end{aligned}
\]
\[
\begin{aligned}
P_{k,t} & = Q_{k,t} + A_{k,t}^{T} P_{k,t+1} A_{k,t} \\
& + \sum_{i,j=1}^{p} \gamma_{ij}^{k} (C_{k,t})^{T} P_{k,t+1} C_{k,t},
\end{aligned}
\]
\[
\begin{aligned}
P_{k,N} & = G_{k}, \quad P_{k,N} = G_{k}, \quad \ell \in T_k.
\end{aligned}
\]

Proof. From (14), it follows that
\[
\begin{aligned}
\bar{E}_{k} Y_{k}^{k,u,k} & = A_{k} \bar{E}_{k} E_{k} X_{k}^{k,u,k} \\
& + \bar{E}_{k} Y_{k}^{k,u,k} = B_{k} \bar{E}_{k} E_{k} u_{k},
\end{aligned}
\]
$\ell \in T_{k+1},$
\[
\begin{aligned}
\bar{E}_{k} Y_{k}^{k,u,k} = 0.
\end{aligned}
\]

Let $\bar{P}_{k,t} = \bar{P}_{k,t} - P_{k,t}, \ell \in T_k$. By adding to and subtracting
\[
\begin{aligned}
\bar{J}(k, 0; \bar{u}_k) & = \sum_{t=k}^{N-1} \bar{E}_{k} \left[ (Y_{t}^{k,u,k})^{T} \bar{P}_{k,t+1} Y_{t}^{k,u,k} \\
& + (E_{k} Y_{k}^{k,u,k})^{T} \bar{P}_{k,t+1} E_{k} Y_{k}^{k,u,k} \\
& - (E_{k} Y_{k}^{k,u,k})^{T} \bar{P}_{k,t+1} E_{k} Y_{k}^{k,u,k} \\
& - (E_{k} Y_{k}^{k,u,k})^{T} \bar{P}_{k,t+1} E_{k} Y_{k}^{k,u,k} \right]
\end{aligned}
\]
from (13), we have
+ \tilde{u}^T_k \mathbf{R}_{k,k} \tilde{u}_k

= \sum_{\ell=k+1}^{N-1} E_k \left[ (E_k Y^k_{\ell})^T (Q_{k,\ell} + A^T_{k,\ell} P_{k,\ell+1} A_{k,\ell})

+ \sum_{i,j=1}^{p} \gamma_{ij} (C_{k,\ell})^T P_{k,\ell+1} C_{k,\ell}^T - \mathcal{P}_{k,\ell}) E_k Y^k_{\ell} \tilde{u}_k

+ (Y^k_{\ell} \tilde{u}_k - E_k Y^k_{\ell} \tilde{u}_k)^T (Q_{k,\ell} + A^T_{k,\ell} P_{k,\ell+1} A_{k,\ell})\right]

+ \tilde{u}^T_k \mathbf{R}_{k,k} + B^T_k P_{k,k+1} B_k

+ \sum_{i,j=1}^{p} \gamma_{ij} (D^i_{k,k})^T P_{k,k+1} D^j_{k,k} \tilde{u}_k

= \tilde{u}^T_k \left[ \mathbf{R}_{k,k} + B^T_k P_{k,k+1} B_k

+ \sum_{i,j=1}^{p} \gamma_{ij} (D^i_{k,k})^T P_{k,k+1} D^j_{k,k} \right] \tilde{u}_k + \sum_{i,j=1}^{p} \gamma_{ij} (D^i_{k,k})^T P_{k,k+1} D^j_{k,k} \tilde{u}_k. \quad (23)
holds. Here, \( \mathcal{H}_k, \mathcal{W}_k, \beta_k, k \in \mathbb{T}_t \) are given by

\[
\mathcal{W}_k = \mathcal{R}_{k,k} + B_k^T (P_{k,t+1} + T_{k,t+1}) B_{k,k} + \sum_{i,j=1}^p \alpha_{ij}^k (D_{i,k}^T)^T \times (P_{k,k+1} + T_{k,k+1}) D_{j,k},
\]

\[
\mathcal{H}_k = B_k^T (P_{k,k+1} + T_{k,k+1}) A_{k,k} + \sum_{i,j=1}^p \alpha_{ij}^k (D_{i,k}^T)^T \times (P_{k,k+1} + T_{k,k+1}) C_{j,k},
\]

\[
\beta_k = B_k^T \left[ (P_{k,k+1} + T_{k,k+1}) f_{k,k} + \pi_{k,k+1} \right] + \sum_{i,j=1}^p \alpha_{ij}^k (D_{i,k}^T)^T \times (P_{k,k+1} + T_{k,k+1}) d_{j,k} + \rho_{k,k}, \quad k \in \mathbb{T}_t.
\]

with

\[
T_{k,t} = A_k^T T_{k,t+1} A_k + \sum_{i,j=1}^p \gamma_{ij}^k (C_{i,k}^T)^T T_{k,t+1} C_{j,k} - \left\{ A_{i,k}^T T_{k,t+1} B_{i,k} + A_{i,k}^T T_{k,t+1} B_{i,k} \right\} + \sum_{i,j=1}^p \gamma_{ij}^k (C_{i,k}^T)^T P_{k,t+1} D_{j,k} + (C_{i,k}^T)^T T_{k,t+1} D_{j,k}, \quad k \in \mathbb{T}_t,
\]

\[
T_{k,N} = 0, \quad T_{k,N} = 0, \quad k \in \mathbb{T}_t,
\]

and

\[
\pi_{k,t} = A_k^T P_{k,t+1} (f_{k,t} - B_{k,k} W_{k}^j \beta_k) + A_k^T T_{k,t+1} (f_{k,t} - B_{k,k} W_{k}^j \beta_k) + \sum_{i,j=1}^p \gamma_{ij}^k (C_{i,k}^T)^T P_{k,t+1} D_{j,k} + (C_{i,k}^T)^T T_{k,t+1} D_{j,k} + \pi_{k,t}^T T_{k,t+1} + q_{k,k}, \quad k \in \mathbb{T}_t.
\]

Furthermore, in (29) \( \text{Ran}(\mathcal{W}_k) \) is the range of \( \mathcal{W}_k \), and \( X_{k,t,x}^{t,x,*} \) is computed via

\[
X_{k+1}^{t,x,*} = \left[ (A_{k,k} - B_{k,k} W_{k}^j \mathcal{H}_k) X_{k}^{t,x,*} - B_{k,k} W_{k}^j \beta_k + f_{k,k} \right] + \sum_{i,j=1}^p \gamma_{ij}^k (C_{i,k}^T)^T P_{k,t+1} D_{j,k} + (C_{i,k}^T)^T T_{k,t+1} D_{j,k} + \pi_{k,t}^T T_{k,t+1} + q_{k,k}, \quad k \in \mathbb{T}_t.
\]

Under any of the above conditions, \( u_{k,t,x}^{t,x,*} \) in (17) is selected as

\[
u_{k,t,x}^{t,x,*} = -W_{k}^j \mathcal{H}_k X_{k}^{t,x,*} - W_{k}^j \beta_k, \quad k \in \mathbb{T}_t.
\]
Above theorem is concerned with the existence of open-loop equilibrium control. Another important issue is the uniqueness of open-loop equilibrium control, which is studied in the following theorem.

**Theorem 3.5:** The following statements are equivalent.

1. Problem (LQ)$_{t_x}$ admits a unique open-loop equilibrium control.
2. The following assertions hold.
   - a) The condition (25) is satisfied.
   - b) $\mathcal{W}_k, k \in \mathbb{T}_t$, are invertible, where $\mathcal{W}_k$ is given in (30).

Under any of the above conditions, the unique open-loop equilibrium control is given by

$$u_{t,x}^i = -\mathcal{W}_k^{-1} \mathcal{H}_k X_{t,x}^i - \mathcal{W}_k^{-1} \beta_k, \ k \in \mathbb{T}_t$$

with $X_{t,x}^i$ given by

$$X_{t,x}^i = \begin{cases} 
(A_k - B_k \mathcal{W}_k^{-1} \mathcal{H}_k) X_{t,x}^{i-1} \\
- B_k \mathcal{W}_k^{-1} \beta_k + f_k, k \in \mathbb{T}_t \\
\sum_{i=1}^{p} \left[ (C_k - D_k \mathcal{W}_k^{-1} \mathcal{H}_k) X_{t,x}^{i-1} - D_k \mathcal{W}_k^{-1} \beta_k + d_{k,i} \right] u_i, \\
x_t \in \mathbb{T}_t.
\end{cases}$$

**Proof.** i) $\Rightarrow$ ii). The condition (25) naturally holds. We further have b). Otherwise, controls of form (37) are also open-loop equilibrium control.

ii) $\Rightarrow$ i). According to Theorem 3.1 and Theorem 3.4, Problem (LQ)$_{t_x}$ admits an open-loop equilibrium control. Due to the nonsingularity of $\mathcal{W}_k, k \in \mathbb{T}_t$, and the proof of Theorem 3.3, the open-loop equilibrium control is unique, which is given by (38). \(\square\)

**IV. THE CASE WITH ALL THE INITIAL PAIRS**

In this section, we will let the initial time $t$ and initial state $x$ range over $\mathbb{T}$ and $l^2_{F}(t; \mathbb{R}^n)$, respectively; this is referred to as the case with all the initial pairs. Problem (LQ) for the initial pair $(t, x)$ will be simply denoted as Problem (LQ)$_{t_x}$, and similar meanings hold for other initial pairs.

Firstly, we give an interesting result on the unique existence of open-loop equilibrium control, which follows from Theorem 3.5.

**Proposition 4.1:** Let $t \in \mathbb{T}$ and $x \in l^2_{F}(t; \mathbb{R}^n)$. Then, the following statements are equivalent.

1. Problem (LQ)$_{t_x}$ admits a unique open-loop equilibrium control.
2. For any $k \in \mathbb{T}_t$ and any $\xi \in l^2_{F}(k; \mathbb{R}^n)$, Problem (LQ)$_{k \xi}$ admits a unique open-loop equilibrium control.

Unfortunately, result similar to Proposition 4.1 does not hold if we just consider the existence of open-loop equilibrium control. Alternatively, the following assertion holds.

**Theorem 4.1:** The following statements are equivalent.

1. Problem (LQ)$_{t_x}$ admits an open-loop equilibrium control.
2. The set of constrained LDEs

$$\begin{align*}
& P_{k,t} = Q_{k,t} + A^T_{k,t} P_{k,t+1} A_{k,t} \\
& \quad + \sum_{i=1}^{p} \gamma_{k,t}^{ij} (C_{k,t}^i)^T P_{k,t+1} C_{k,t}^j \\
& P_{k,t} = Q_{k,t} + A^T_{k,t} P_{k,t+1} A_{k,t} \\
& \quad + \sum_{i=1}^{p} \gamma_{k,t}^{ij} (C_{k,t}^i)^T P_{k,t+1} C_{k,t}^j, \\
P_{k,N} = G_k, \quad & P_{k,N} = G_k, \quad t \in \mathbb{T}_t, \\
& k \in \mathbb{T}, \\
& \mathcal{W}_k \succeq 0,
\end{align*}\tag{39}$$

and the set of constrained GDREs

$$\begin{align*}
& T_{k,t} = A^T_{k,t} T_{k,t+1} A_{k,t} \\
& \quad + \sum_{i=1}^{p} \gamma_{k,t}^{ij} (C_{k,t}^i)^T T_{k,t+1} C_{k,t}^j \\
& T_{k,t} = A^T_{k,t} T_{k,t+1} A_{k,t} \\
& \quad + \sum_{i=1}^{p} \gamma_{k,t}^{ij} (C_{k,t}^i)^T T_{k,t+1} C_{k,t}^j, \\
& T_{k,N} = 0, \quad & T_{k,N} = 0, \\
& \ell \in \mathbb{T}_t, \\
& \mathcal{W}_k T_{k \ell}^i \mathcal{H}_k - \mathcal{H}_k = 0, \quad k \in \mathbb{T},
\end{align*}\tag{40}$$

and the set of constrained LDEs

$$\begin{align*}
& \pi_{k,t} = A^T_{k,t} \pi_{k,t+1} (f_k - B_k \mathcal{W}_k^j \beta_k) \\
& \quad + A^T_{k,t} T_{k,t+1} (f_k - B_k \mathcal{W}_k^j \beta_k) \\
& \quad + \sum_{i=1}^{p} \gamma_{k,t}^{ij} \left[ (C_k^i)^T P_{k,t+1} D_k^i \right] \\
& \quad \times T_{k,t+1} (D_k^j - D_k^j \mathcal{W}_k^j \beta_k + (C_k^j)^T) \\
& \quad + A^T_{k,t} \pi_{k,t+1} + g_k, k \in \mathbb{T}_t \\
& \pi_{k,N} = g_k, \\
& \mathcal{W}_k \mathcal{W}_k^j \beta_k - \beta_k = 0, \quad k \in \mathbb{T}
\end{align*}\tag{41}$$

are solvable in the sense that

$$\begin{align*}
& \mathcal{W}_k \succeq 0, \\
& \mathcal{W}_k \mathcal{W}_k^j \mathcal{H}_k - \mathcal{H}_k = 0, \\
& \mathcal{W}_k \mathcal{W}_k^j \beta_k - \beta_k = 0, \quad \ell \in \mathbb{T}_t, \\
& k \in \mathbb{T}, \quad k \succeq 0.
\end{align*}\tag{42}$$
holds, i.e., the solutions of (39)-(41) satisfy (42). Here,

\[
\mathcal{W}_k = R_{k,k} + B^T_{k,k}P_{k,k+1}B_{k,k} + \sum_{j=1}^{p} \gamma_{k,k}^j (D^i_{k,k})^T P_{k,k+1} D^i_{k,k},
\]

\[
\mathcal{W}_k = R_{k,k} + B^T_{k,k} (P_{k,k+1} + T_{k,k+1}) B_{k,k} + \sum_{j=1}^{p} \gamma_{k,k}^j (D^i_{k,k})^T (P_{k,k+1} + T_{k,k+1}) D^i_{k,k},
\]

\[
\mathcal{H}_k = B^T_{k,k} (P_{k,k+1} + T_{k,k+1}) A_{k,k}
\]

\[
+ \sum_{j=1}^{p} \gamma_{k,k}^j (D^i_{k,k})^T (P_{k,k+1} + T_{k,k+1}) C^i_{k,k},
\]

\[
+ \gamma_{k,k}^1 (D^i_{k,k})^T (P_{k,k+1} + T_{k,k+1}) A^1_{k,k},
\]

\[
+ \rho_{k,k},
\]

\[
k \in \mathbb{T}.
\]

Under any of the above conditions, control of the following form (37) is an open-loop equilibrium control of Problem (LQ)$_{tx}$.

Proof: i)⇒ii). From Theorem 3.4, the constrained LDEs (39) are solvable, and for any \( t \in \mathbb{T}, x \in \mathcal{I}_1^2(t; \mathbb{R}^n) \), the condition (29) holds, i.e.,

\[
\mathcal{H}_t x^t + \beta_t \in \text{Ran} (\mathcal{W}_t), \quad k \in \mathbb{T}_t.
\]

Especially, we have

\[
\mathcal{H}_t x + \beta_t \in \mathcal{W}_t, \quad t \in \mathbb{T},
\]

equivalently,

\[
\mathcal{W}_t W^t (\mathcal{H}_t x + \beta_t) = \mathcal{H}_t x + \beta_t, \quad t \in \mathbb{T}.
\]

Let \( x = 0 \) in (43), we have

\[
\mathcal{W}_t W^t (\mathcal{H}_t x + \beta_t) = \mathcal{H}_t x + \beta_t, \quad t \in \mathbb{T},
\]

which further implies

\[
\mathcal{W}_t W^t \mathcal{H}_t x = \mathcal{H}_t x, \quad t \in \mathbb{T}.
\]

Noting that (44) holds for any \( t \in \mathcal{I}_2^2(t; \mathbb{R}^n) \), we obtain

\[
\mathcal{W}_t W^t \mathcal{H}_t x = \mathcal{H}_t x, \quad t \in \mathbb{T}.
\]

Hence, (40) and (41) are solvable.

ii)⇒i). As (39), (40) and (41) are solvable and by Theorem 3.4, for any \( t \in \mathbb{T} \) and any \( x \in \mathcal{I}_2^2(t; \mathbb{R}^n) \) Problem (LQ)$_{tx}$ admits an open-loop equilibrium control.

Corollary 4.1: Let

\[
Q_{k,k}, \bar{Q}_{k,k} \succeq 0, \quad R_{k,k}, \bar{R}_{k,k} > 0, \quad k \in \mathbb{T}, \quad \ell \in \mathbb{T}_k.
\]

Then, the following statements are equivalent:

i) For any \( t \in \mathbb{T} \) and any \( x \in \mathcal{I}_2^2(t; \mathbb{R}^n) \), Problem (LQ)$_{tx}$ admits an open-loop equilibrium control.

ii) (40) and (41) are solvable.

Proof. In this situation, \( \mathcal{W}_k, k \in \mathbb{T} \) are positive definite, i.e., \( \mathcal{W}_k \succ 0, k \in \mathbb{T} \). Hence, the conclusion follows. □

Let us take some rough observations. Assuming (45), consider Problem (LQ)$_{tx}$ for \( t \in \mathbb{T} \) and \( x \in \mathcal{I}_2^2(t; \mathbb{R}^n) \). Let us begin with \( t = N - 1 \). Noting that \( \mathcal{W}_{N-1} = \mathcal{W}_{N-1} > 0 \), Problem (LQ)$_{N-1, x}$ admits a unique open-loop equilibrium control and \( u^1_{N-1, x} \) is easily obtained

\[
u^1_{N-1, x} = -\mathcal{W}^{-1}_{N-1} \mathcal{H}_{N-1} x - \mathcal{W}^{-1}_{N-1} \beta_{N-1}.
\]

Now move to the case \( t = N - 2 \). If we have selected \( u^1_{N-2, x} \), from Lemma 3.1 and Lemma 3.2 we have

\[
J(N-2, x; (u^1_{N-2, x})) = u^T_{N-2} \bar{W}_{N-2} u_{N-2} \nonumber
\]

\[
+ 2 \{ \rho_{N-2, N-2} + B^T_{N-2, N-2} \bar{E}_{N-2} u_{N-2} \}
\]

\[
+ \sum_{i=1}^{p} (D^i_{N-2, N-2})^T \bar{E}_{N-1} (Z^1_{N-1, -2})^T u_{N-2}
\]

\[
+ J(N-2, x; (0, u^1_{N-2, x})),
\]

\[
\Delta (\bar{W}_{N-2} u_{N-2}, u_{N-2}) + 2(M_{N-2} (Z^1_{N-1, -2}), u_{N-2})
\]

\[
+ J(N-2, x; (0, u^1_{N-2, x})).
\]

In the above, \( Z^1_{N-1, -2} \) is computed via

\[
\begin{align*}
Z^1_{N-1, -2} &= A^T_{N-2, N-1} X^1_{N-1, N-2} + A^T_{N-2, N-1} X^1_{N-1, N-2} \nonumber
\end{align*}
\]

\[
+ \sum_{i=1}^{p} \left[ (C^i_{N-2, N-1})^T E_{N-2} (Z^1_{N-2, N-1})^T e^i_{N-2, N-1} \right]
\]

\[
+ \sum_{i=1}^{p} (D^i_{N-2, N-1})^T E_{N-2} (Z^1_{N-2, N-1})^T u^1_{N-2, x} \nonumber
\]

\[
+ \rho_{N-2, N-1},
\]

\[
Z^1_{N-2, N-2} = G^T_{N-2, N-1} X^1_{N-2, N-2} + G^T_{N-2, N-1} X^1_{N-2, N-2} \nonumber
\]

\[
+ \rho_{N-2, N-1},
\]

\[
\ell \in \{ N-2, N-1 \},
\]

where \( X^1_{N-1, -2} \) is given by

\[
\begin{align*}
X^1_{N-1, -2} &= (A_{N-2, N-1} X^1_{N-1, N-2} + A_{N-2, N-1} X^1_{N-1, N-2}) \nonumber
\end{align*}
\]

\[
+ \sum_{i=1}^{p} \left[ (C^i_{N-2, N-1})^T E_{N-2} (Z^1_{N-2, N-1})^T e^i_{N-2, N-1} \right]
\]

\[
+ \sum_{i=1}^{p} (D^i_{N-2, N-1})^T E_{N-2} (Z^1_{N-2, N-1})^T u^1_{N-2, x} \nonumber
\]

\[
+ \rho_{N-2, N-1},
\]

\[
\ell \in \{ N-2, N-1 \},
\]

and \( \langle \cdot, \cdot \rangle \) is the inner product on \( \mathbb{R}^m \), and

\[
M_{N-2} (Z^1_{N-1, -2}) = \rho_{N-2, N-2} + B^T_{N-2, N-2} \bar{E}_{N-2} Z^1_{N-1, -2}
\]

\[
+ \sum_{i=1}^{p} (D^i_{N-2, N-2})^T \bar{E}_{N-1} (Z^1_{N-1, -2})^T u^1_{N-2, x}.
\]

If we “select”

\[
u^1_{N-2, x} = -\mathcal{W}^{-1}_{N-2} M_{N-2} (Z^1_{N-1, -2}),
\]

\[
(48)
\]
then the following inequality
\[ J(N - 2, x; (u_{N-2}^N, x^N, u_{N-1}^N, x^N)) \leq J(N - 2, x; (u_{N-2}^N, u_{N-1}^N)) \]
seems to hold.

However, it should be mentioned that it is questionable about (48). If \( u_{N-2}^N, x^N \) exists, we should have
\[ u_{N-1}^N = -W_{N-1}^{-1}H_{N-1}X_{N-1}^N - W_{N-1}^{-1}\beta_{N-1} \]
and
\[ X_{N-1}^N = A_{N-2,N-2} x + B_{N-2,N-2} u_{N-2}^N + \sum_{i=1}^{P} (C_i^2, 2N-2)x u_{N-2}^N \]
\[ + \sum_{i=1}^{P} (D_i^2, 2N-2) u_{N-2}^N \]
Hence, \( u_{N-2}^N, x^N \) depends on \( u_{N-2}^N, x^N \), and it is so for \( Z_{N-2}^N \). Therefore, the righthand side of (48) is a functional of \( u_{N-2}^N, x^N \), and it cannot be concluded that (48) makes sense under the assumption \( \mathbb{W}_{N-2} \). Recall that \( \{\{T_{k,\ell}, \ell \in T_k \}, k \in T\} \) is also needed to characterize the open-loop equilibrium control, and that for \( k \in T \)
\[ \mathbb{W}_k = \mathbb{W}_k + B_{T_k, k+1}^T B_{k, k+1} + \sum_{i,j=1}^{P} (D_i^2, k+1) T_{k, k+1} D_i^2, k, k+1. \]
Note that elements in \( \{\{T_{k,\ell}, \ell \in T_k \}, k \in T\} \) are generally nonsymmetric. So far, it is not known now whether or not \( \mathbb{W}_k \) could ensure the nonsingularity of \( \mathbb{W}_k \). Therefore, we have to check case by case the solvability of (40) (41) (by validating (42)).

V. MULTI-PERIOD MEAN-VARIANCE PORTFOLIO SELECTION

Consider a capital market consisting of one riskless asset and \( n \) risky assets within a time horizon \( N \). Let \( s_k > 1 \) be a given deterministic return of the riskless asset at time period \( k \) and \( e_k = (e_k^1, \ldots, e_k^n)^T \) the vector of random returns of the \( n \) risky assets at period \( k \). We assume that vectors \( e_k \), \( k = 0, 1, \ldots, N - 1 \), are statistically independent and the only information known about the random return vector \( e_k \) is its first two moments: its mean \( \mathbb{E}(e_k) = (\mathbb{E}e_k^1, \mathbb{E}e_k^2, \ldots, \mathbb{E}e_k^n)^T \) and its covariance \( \text{Cov}(e_k) = \mathbb{E}((e_k - \mathbb{E}e_k)(e_k - \mathbb{E}e_k)^T) \).
Clearly, \( \text{Cov}(e_k) \) is nonnegative definite, i.e., \( \text{Cov}(e_k) \succeq 0 \).

Let \( X_k \) be the wealth of the investor at the beginning of the \( k \)-th period, and let \( u_k^i, i = 1, 2, \ldots, n, \) be the amount invested in the \( i \)-th risky asset at period \( k \). Then \( X_k - \sum_{i=1}^{n} u_k^i \) is the amount invested in the riskless asset at period \( k \), and the wealth at the beginning of the \( (k + 1) \)-th period [22] is given by
\[ X_{k+1} = \sum_{i=1}^{n} e_k^i u_k^i + \left( X_k - \sum_{i=1}^{n} u_k^i \right) s_k = s_k X_k + O_k^T u_k, \]
where \( O_k \) is the excess return vector of risky assets [22] defined as
\[ O_k = (O_k^1, O_k^2, \ldots, O_k^n)^T \]
\[ = (e_k^1 - s_k, e_k^2 - s_k, \ldots, e_k^n - s_k)^T. \]

Clearly, \( X_k \in \mathbb{R}, k \in T \). In this section, we consider the case where short-selling of stocks is allowed, i.e., \( u_k^i, i = 1, \ldots, k \), could take values in \( \mathbb{R} \), which leads to an unconstrained mean-variance portfolio selection formulation.

Let
\[ \mathcal{F}_k = \sigma(e_k, \ell = 0, 1, \ldots, k - 1), \]
which contains \( \mathcal{F}_k' = \sigma(X_k, \ell = 0, 1, \ldots, k) \). Then, the time-inconsistent version of multi-period mean-variance problem [22] can be formulated as follows:

**Problem (MV).** Letting \( t \in T \) and \( x \in \mathbb{I}_2^T(t; \mathbb{R}^n) \), find \( u^* \in \mathbb{I}_2^T(t; \mathbb{R}^n) \) such that
\[ J_m(t, x; u^*) = \inf_{u \in \mathbb{I}_2^T(t; \mathbb{R}^n)} J_m(t, x; u). \]

Here,
\[ J_m(t, x; u) = \lambda \mathbb{E}(X_N - \mathbb{E}X_N)^2 - \mathbb{E}X_N, \]
which is subject to
\[ \begin{aligned}
X_{k+1} &= s_k X_k + O_k^T u_k, \\
X_k &= x
\end{aligned} \]
with \( \lambda > 0 \) the trade-off parameter between the mean and the variance of the terminal wealth.

It is noted that some nondegenerate assumptions are posed in [2], [6], [9], [10], [17], [18]. Specifically, the volatilities of the stocks in [2], [6], [17], [18] and the return rates of the risky securities in [9], [10], [22] are assumed to be nondegenerate. In this section, we do not pose the nondegenerate constraint on \( \text{Cov}(e_k), \text{Cov}(O_k) \), \( k \in T \), and want to see what is the weakest condition on the existence of open-loop equilibrium portfolio control of Problem (MV).

To solve Problem (MV), we shall transform (49) into a linear controlled system of form (4), by which the general theory in above sections will work. Precisely, define
\[ \begin{aligned}
w_k^i &= e_k^i - s_k - \mathbb{E}(e_k^i - s_k), \\
D_k^i &= (0, 0, 0, 0, 0, \ldots), \\
i &= 1, 2, \ldots, n, \quad k = 0, 1, \ldots, N - 1,
\end{aligned} \]
where the \( i \)-th entry of \( D_k^i \) is 1. Then, \( \{w_k = (w_k^1, \ldots, w_k^n)^T, k \in T\} \) is a martingale difference sequence as \( e_k, k = 1, \ldots, N - 1 \), are statistically independent. Furthermore,
\[ \mathbb{E}k[w_k^i w_k^j]^T = \mathbb{E}[w_k^i w_k^j]^T = \text{Cov}(e_k) = (\gamma_k^i)^{n \times n}. \]

This leads to
\[ \begin{aligned}
X_{k+1} &= (s_k X_k + \mathbb{E}(O_k)^T u_k) + \sum_{i=1}^{n} D_k^i u_k w_k^i, \\
X_k &= x.
\end{aligned} \]

Due to Theorem 4.1, we have the following result.

**Theorem 5.1:** The following statements are equivalent.

i) For any \( t \in T \) and any \( x \in \mathbb{I}_2^T(t; \mathbb{R}) \), Problem (MV) admits an open-loop equilibrium portfolio control.

ii) \( \mathbb{E}k(\text{Cov}(O_k), k \in T \).

Under any of the above conditions,
\[ u_k^{x, \ast} = -W_k^i \beta_k, \quad k \in T_i. \]
This is an open-loop equilibrium portfolio control for the initial pair \((t, x)\), where
\[
\begin{align*}
W_k &= P_{k+1}\text{Cov}(O_k), \\
\beta_k &= \pi_{k+1}E O_k, \\
\end{align*}
\]  
with
\[
\begin{align*}
P_k &= s_k^2 P_{k+1}, \\
P = s_k^2 P_{k+1} \equiv 0, \\
P_N = \lambda, \\
\end{align*}
\]  
and
\[
\begin{align*}
W_k &= \sum_{i,j=1}^n \gamma_{ij} (D_j^i)^T P_{k+1} D_j^i, \\
\beta_k &= \pi_{k+1} E O_k, \\
\end{align*}
\]  
with \(W_k, \beta_k, k \in \mathbb{T}\) given in (53).

\textbf{Proof.} The proof follows from Theorem 3.5, Proposition 4.1 and Theorem 5.1.

Note that \(\text{Cov}(O_k) > 0, k \in \mathbb{T}\) is a common assumption in multi-period mean-variance portfolio selection [9] [11] [22]. In this situation, the open-loop equilibrium portfolio control for the initial pair \((t, x)\) is
\[
\begin{align*}
u_{t,x}^{t,*} &= \frac{1}{2\lambda s_{k+1} \cdots s_{N-1}}(\text{Cov}(O_k))^{-1} E O_k, \\
k \in \mathbb{T},
\end{align*}
\]  
This section just studied the simplest dynamic mean-variance model [22]. In the future, dynamic mean-variance portfolio optimizations are much desirable for the more general models.

\section{VI. Conclusion}

In this paper, the open-loop time-consistent equilibrium control is investigated for a kind of mean-field stochastic LQ problem, where both the system matrices and the weighting matrices are depending on the initial time, and the conditional expectations of the control and state enter quadratically into the cost functional. Necessary and sufficient conditions are presented for both the case with a fixed initial pair and the case with all the initial pairs. Furthermore, a set of constrained GDEs and two sets of constrained LDEs are introduced to characterize the open-loop equilibrium control. Note that this paper is concerned with the time-consistency of open-loop control. For future research, the time-consistency of the strategy should be studied.

\section{References}

APPENDIX

A. Proof of Lemma 3.1

Let us replace $u_k$ with $u_k + \lambda \bar{u}_k$ in the forward SDE of (15), and denote its solution by $X^{k,\lambda}_\ell$. Then, we have

$$
\frac{X^{k,\lambda}_\ell - X^{k,-\lambda}_\ell}{\lambda} = \left( A_{k,\ell} \frac{X^{k,\lambda}_\ell - X^{k,-\lambda}_\ell}{\lambda} + \bar{A}_{k,\ell} \frac{\mathbb{E}_k X^{k,\lambda}_\ell - \mathbb{E}_k X^{k,-\lambda}_\ell}{\lambda} \right) + \sum_{i=1}^{P} \left( C_{k,\ell} \frac{X^{k,\lambda}_\ell - X^{k,-\lambda}_\ell}{\lambda} + \bar{C}_{k,\ell} \frac{\mathbb{E}_k X^{k,\lambda}_\ell - \mathbb{E}_k X^{k,-\lambda}_\ell}{\lambda} \right) w^i_{\ell},
$$

where

$$
X^{k,\lambda}_{\ell+1} = A_{k,\ell} X^{k,\lambda}_\ell + \bar{A}_{k,\ell} \mathbb{E}_k X^{k,\lambda}_\ell + \sum_{i=1}^{P} (C_{k,\ell} X^{k,\lambda}_\ell + \bar{C}_{k,\ell} \mathbb{E}_k X^{k,\lambda}_\ell) w^i_{\ell},
$$

and

$$
X^{k,-\lambda}_{\ell+1} = A_{k,\ell} X^{k,-\lambda}_\ell + \bar{A}_{k,\ell} \mathbb{E}_k X^{k,-\lambda}_\ell + \sum_{i=1}^{P} (C_{k,\ell} X^{k,-\lambda}_\ell + \bar{C}_{k,\ell} \mathbb{E}_k X^{k,-\lambda}_\ell) w^i_{\ell}.
$$

We can verify this equality by induction on $\ell$. Here, we have used the fact $\mathbb{E}_k u_k = u_k$. Note that $X^{k,\lambda}_{\ell} = X^{k,u}_{\ell} + \lambda Y^{k,\lambda}_{\ell}, \forall \ell \in \mathbb{T}_k$. Then, we have

$$
J(k, \zeta; (u_k + \lambda \bar{u}_k, \mathbb{E}_k u_k | \mathbb{T}_{k+1})) = J(k, \zeta; u)
$$

for $\ell = 1, \ldots, N-1$.

Denoting $Y^{k,\lambda}_{\ell}$ by $Y^{k,\lambda}_{k,\ell}$, we get

$$
\begin{align*}
Y^{k,\lambda}_{\ell+1} &= A_{k,\ell} Y^{k,\lambda}_{\ell} + \bar{A}_{k,\ell} \mathbb{E}_k Y^{k,\lambda}_{\ell} + \sum_{i=1}^{P} (C_{k,\ell} Y^{k,\lambda}_{\ell} + \bar{C}_{k,\ell} \mathbb{E}_k Y^{k,\lambda}_{\ell}) w^i_{\ell}, \\
Y^{k,\lambda}_{k,\ell} &= 0, \quad \ell \in \mathbb{T}_{k+1}.
\end{align*}
$$

Here, we have used the fact $\mathbb{E}_k u_k = u_k$. Note that $X^{k,\lambda}_{\ell} = X^{k,u}_{\ell} + \lambda Y^{k,\lambda}_{\ell}, \forall \ell \in \mathbb{T}_k$. Then, we have

$$
J(k, \zeta; (u_k + \lambda \bar{u}_k, \mathbb{E}_k u_k | \mathbb{T}_{k+1})) = J(k, \zeta; u)
$$

for $\ell = 1, \ldots, N-1$.

Denoting $Y^{k,\lambda}_{\ell}$ by $Y^{k,\lambda}_{k,\ell}$, we get

$$
\begin{align*}
Y^{k,\lambda}_{\ell+1} &= A_{k,\ell} Y^{k,\lambda}_{\ell} + \bar{A}_{k,\ell} \mathbb{E}_k Y^{k,\lambda}_{\ell} + \sum_{i=1}^{P} (C_{k,\ell} Y^{k,\lambda}_{\ell} + \bar{C}_{k,\ell} \mathbb{E}_k Y^{k,\lambda}_{\ell}) w^i_{\ell}, \\
Y^{k,\lambda}_{k,\ell} &= 0, \quad \ell \in \mathbb{T}_{k+1}.
\end{align*}
$$

Here, we have used the fact $\mathbb{E}_k u_k = u_k$. Note that $X^{k,\lambda}_{\ell} = X^{k,u}_{\ell} + \lambda Y^{k,\lambda}_{\ell}, \forall \ell \in \mathbb{T}_k$. Then, we have

$$
J(k, \zeta; (u_k + \lambda \bar{u}_k, \mathbb{E}_k u_k | \mathbb{T}_{k+1})) = J(k, \zeta; u)
$$

for $\ell = 1, \ldots, N-1$.
\[+ \mathbb{E}_k \left[ u_k^T (R_{k,k} + \bar{R}_{k,k}) \bar{u}_k \right]
+ \mathbb{E}_k \left[ (Y_{k,k} \bar{u}_k) G_k Y_{k,k} \bar{u}_k \right]
+ (E_k Y_{k,k})^T \bar{G}_k \mathbb{E}_k Y_{k,k} \bar{u}_k \right]. \]

(60)

Form (15), it holds that \( \mathbb{E}_k Z_{k}^N = G_k \mathbb{E}_k X_{N}^N + g_k \) and \( Z_{k}^N = \mathbb{E}_k Z_{k}^N - \mathbb{E}_k Z_{k}^N = G_k (X_{N}^N - \mathbb{E}_k X_{N}^N) \). Noting \( Y_{k,k} \bar{u}_k = 0 \), then, we have

\[
\sum_{\ell=k}^{N-1} \mathbb{E}_k \left[ (X_{k}^\ell)^T Q \ell \bar{u}_k + q_{k,\ell} \bar{u}_k \right]
+ [\mathbb{E}_k X_{k}^T \bar{Q}_k \ell \mathbb{E}_k Y_{k,k}] + u_k^T (R_{k,k} + \bar{R}_{k,k}) \bar{u}_k
+ \rho_{k,\ell}^T \bar{u}_k + \mathbb{E}_k \left[ (X_{k,\ell}^T \bar{Q}_k \ell \mathbb{E}_k Y_{k,k}) \bar{u}_k \right]
+ [\mathbb{E}_k X_{k,\ell} Y_{k,k}] \bar{u}_k + \mathbb{E}_k [s_1^T Y_{k,k}] \bar{u}_k
= \sum_{\ell=k}^{N-1} \mathbb{E}_k \left[ (Q \ell k \bar{u}_k - \mathbb{E}_k X_{k,k}^T \bar{u}_k) \right]
+ A_{k,k}^T \mathbb{E}_k Z_{k+1}^N - \mathbb{E}_k Z_{k+1}^N
+ \sum_{i=1}^{p} (C_{k,i}^T \mathbb{E}_k (Z_{k+1}^N w_{i}^T) - \mathbb{E}_k (Z_{k+1}^N w_{i}^T))
+ (Z_{k}^N - \mathbb{E}_k Z_{k}^N)^T (Y_{k,k} \bar{u}_k - \mathbb{E}_k Y_{k,k} \bar{u}_k)
+ \mathbb{E}_k X_{k}^k \bar{u}_k + q_k \bar{u}_k + A_{k,k}^T \mathbb{E}_k Z_{k+1}^N
+ \sum_{i=1}^{p} (C_{k,i}^T \mathbb{E}_k (Z_{k+1}^N w_{i}^T) - \mathbb{E}_k Z_{k}^N)^T \mathbb{E}_k Y_{k,k} \bar{u}_k\]
+ \left[ \mathbb{R}_{k,k} \bar{u}_k + B_{k,k}^T \mathbb{E}_k Z_{k+1}^N \right]
+ \sum_{i=1}^{p} (D_{k,i}^T \mathbb{E}_k (Z_{k+1}^N w_{i}^T) + \rho_k \bar{u}_k)^T \bar{u}_k
= \left[ \mathbb{R}_{k,k} \bar{u}_k + B_{k,k}^T \mathbb{E}_k Z_{k+1}^N \right]
+ \sum_{i=1}^{p} (D_{k,i}^T \mathbb{E}_k (Z_{k+1}^N w_{i}^T) + \rho_k \bar{u}_k)^T \bar{u}_k.

This together with (60) implies the conclusion. \( \square \)

B. Proof of Theorem 3.1

i)⇒ii). Let \( u_{t,x}^*, \) be an open-loop equilibrium control. As (18) is a decoupled FBS\(A\)E, (18) is solvable. From (12) we have

\[
J(k, X_{k}^t, x^*, (u_{t,x}^*, \lambda \bar{u}_k, u_{t,x}^* | \bar{v}_{k+1})) - J(k, X_{k}^t, x^*, u_{t,x}^*)
= 2\lambda \left[ \mathbb{R}_{k,k} \bar{u}_k + B_{k,k}^T \mathbb{E}_k Z_{k+1}^N + \sum_{i=1}^{p} (D_{k,i}^T \mathbb{E}_k (Z_{k+1}^N w_{i}^T) + \rho_k \bar{u}_k)^T \bar{u}_k \right]
+ \lambda^2 \tilde{J} (k, 0; \bar{u}_k)
\ge 0.
\]

(61)

Noting that (61) holds for any \( \lambda \in \mathbb{R} \) and \( \bar{u}_k \in L^2_{\mathcal{F}}(\mathcal{K}; \mathbb{R}^m) \), we have (16) and (17). In fact, if (16) was not satisfied, then there would be a \( \bar{u}_k \) such that \( \lim \lambda \to \infty J(k, X_{k}^t, x^*, u_{t,x}^*, \lambda \bar{u}_k) = -\infty \). This is impossible. Furthermore, if for some \( k_0 \in T_t \)

\[
\gamma_{k_0} = \mathbb{R}_{k_0,k_0} u_{t,x}^* + B_{k_0,k_0}^T \mathbb{E}_k Z_{k+1}^N w_{k_0}^i + \rho_{k_0,k_0}
\ne 0,
\]

we let \( \bar{u}_k = \gamma_{k_0} \). Then, (61) implies that

\[
2\lambda |\gamma_{k_0}|^2 + \lambda^2 \tilde{J} (k_0, 0; \gamma_{k_0}) \ge 0
\]

holds for any \( \lambda \in \mathbb{R} \). However, for negative number \( \lambda \) with sufficient small magnitude, it holds that

\[
2\lambda |\gamma_{k_0}|^2 + \lambda^2 \tilde{J} (k_0, 0; \gamma_{k_0}) < 0,
\]

and contradiction arises. Therefore, \( \gamma_{k_0} \) must be 0, and (17) holds.

ii)⇒i). In this case, for any \( \lambda \in \mathbb{R} \) and \( \bar{u}_k \in L^2_{\mathcal{F}}(\mathcal{K}; \mathbb{R}^m) \) we have

\[
J(k, X_{k}^t, x^*, (u_{t,x}^*, \lambda \bar{u}_k, u_{t,x}^* | \bar{v}_{k+1})) - J(k, X_{k}^t, x^*, u_{t,x}^*) = \lambda^2 \tilde{J} (k, 0; \bar{u}_k)
\ge 0.
\]

Hence, \( u_{t,x}^* \) is an open-loop equilibrium control. \( \square \)

C. Proof of Lemma 3.3

It is assumed that \( u_{t,x}^* = \Psi_x X_{k}^t, x^*, \alpha_{t,x} \in T_{T} \). Then, we have

\[
X_{k}^{N,t,x} = A_{k,N-1} X_{N-1}^{k,t,x} + A_{k,N-1} \mathbb{E}_k X_{N-1}^{k,t,x}
+ B_{k,N-1} \Psi_{N-1} X_{N-1}^{t,x} + B_{k,N-1} \mathbb{E}_k X_{N-1}^{t,x}
+ B_{k,N-1} \alpha_{N-1} + f_{k,N-1}
+ \sum_{i=1}^{p} \left( C_{k,i}^{N-1} X_{N-1}^{t,x} + C_{k,i}^{N-1} \mathbb{E}_k X_{N-1}^{t,x}
+ D_{k,i}^{N-1} \Psi_{N-1} E_k X_{N-1}^{t,x}
+ D_{k,i}^{N-1} \alpha_{N-1} + d_{i,k,N-1} \right) w_{i}^{N-1}.
\]

To calculate \( Z_{N-1}^{k,t,x} \), we need some preparations. Noting that

\[
Z_{k}^{N,t,x} = G_k X_{k}^{N,t,x} + G_k \mathbb{E}_k X_{N-1}^{t,x} + g_k,
\]
we get

\[
A_{k,N-1}^T \mathbb{E}_k X_{N-1}^{k,t,x}
= A_{k,N-1}^T \mathbb{E}_k X_{N-1}^{k,t,x} + G_k \mathbb{E}_k X_{N-1}^{k,t,x} + g_k
+ A_{k,N-1}^T G_k A_{k,N-1} X_{N-1}^{t,x}
+ A_{k,N-1}^T G_k \mathbb{E}_k X_{N-1}^{t,x}
+ A_{k,N-1}^T G_k B_{k,N-1} \Psi_{N-1} X_{N-1}^{t,x}
+ A_{k,N-1}^T G_k B_{k,N-1} \alpha_{N-1} + f_{k,N-1} + A_{k,N-1} g_k.
\]
Therefore,
\[ A_{k,N-1}^T = A_{k-1,N}^T A_{k,N-1} + A_{k,N-1} \psi_{k,N-1} \]
\[ + A_{k,N-1} G_k B_{k,N-1} + A_{k,N-1} G_k A_{k,N-1} - A_{k,N-1} G_k \bar{A}_{k,N-1} \]
\[ + \sum_{i,j=1}^{p} \gamma_{ij} N \left( C_{k,N}^i C_{k,N-1}^j \right)^T G_k C_{k,N-1}^j \]  
\[ \times \bar{E}_k \Xi_{k,t,x} \]
\[ + \sum_{i,j=1}^{p} \gamma_{ij} N \left( C_{k,N-1}^i \right)^T G_k C_{k,N-1}^j \]  
\[ \times \bar{E}_k \Xi_{k,t,x} \]
\[ = \sum_{i,j=1}^{p} \gamma_{ij} N \left( C_{k,N}^i C_{k,N-1}^j \right)^T G_k C_{k,N-1}^j \]
\[ \times \bar{E}_k \Xi_{k,t,x} \]
\[ + \sum_{i,j=1}^{p} \gamma_{ij} N \left( C_{k,N-1}^i \right)^T G_k C_{k,N-1}^j \]  
\[ \times \bar{E}_k \Xi_{k,t,x} \]

Furthermore, it holds that
\[ (C_{k,N-1})^T \Xi_k (Z_{N}^{k,t,x} w_{i-1}^{j}) = \sum_{j=1}^{p} \gamma_{ij} N \left( C_{k,N}^j \right)^T G_k C_{k,N}^j \]
\[ \times \bar{E}_k \Xi_{k,t,x} \]
\[ + \sum_{i,j=1}^{p} \gamma_{ij} N \left( C_{k,N-1}^i \right)^T G_k C_{k,N-1}^j \]  
\[ \times \bar{E}_k \Xi_{k,t,x} \]

Therefore,
\[ Z_{N}^{k,t,x} = \left\{ Q_{k,N-1} + A_{k,N-1} G_k A_{k,N-1} \right\} \]
\[ \times \bar{E}_k \Xi_{k,t,x} \]
\[ + \sum_{i,j=1}^{p} \gamma_{ij} N \left( C_{k,N}^i C_{k,N-1}^j \right)^T G_k C_{k,N-1}^j \]  
\[ \times \bar{E}_k \Xi_{k,t,x} \]
\[ + \sum_{i,j=1}^{p} \gamma_{ij} N \left( C_{k,N-1}^i \right)^T G_k C_{k,N-1}^j \]  
\[ \times \bar{E}_k \Xi_{k,t,x} \]

We now calculate \( Z_{N}^{k,t,x} \). Note that
\[ A_{k,N-2}^T \Xi_k (Z_{N}^{k,t,x} w_{i-1}^{j}) = A_{k,N-2}^T \left[ P_{k,N-1} \Xi_k X_{N-1}^{k,t,x} + T_{k,N-1} \right] \]
\[ \times \bar{E}_k \Xi_{k,t,x} \]

and similar expressions for \( C_{k,N-2}^T \Xi_k (Z_{N}^{k,t,x} w_{i-1}^{j}) \), \( \bar{A}_{k,N-2}^T \Xi_k \Xi_{k,t,x} \), \( \bar{A}_{k,N-2}^T \Xi_k \Xi_{k,t,x} \), and \( \bar{A}_{k,N-2}^T \Xi_k \Xi_{k,t,x} \). Then, from (18) we have
\[ Z_{N}^{k,t,x} = \left\{ Q_{k,N-2} + A_{k,N-2} P_{k,N-1} A_{k,N-2} \right\} \]
\[ \times \bar{E}_k \Xi_{k,t,x} \]
\[ + \sum_{i,j=1}^{p} \gamma_{ij} N \left( C_{k,N-2}^i C_{k,N-2}^j \right)^T P_{k,N-1} C_{k,N-2}^j \]  
\[ \times \bar{E}_k \Xi_{k,t,x} \]
\[ + \sum_{i,j=1}^{p} \gamma_{ij} N \left( C_{k,N-2}^i \right)^T P_{k,N-1} C_{k,N-2}^j \]  
\[ \times \bar{E}_k \Xi_{k,t,x} \]

where
\[ \bar{A}_{k,N-2}^T \Xi_k \Xi_{k,t,x} \]  
\[ = P_{k,N-1} X_{N-1}^{k,t,x} + \bar{P}_{k,N-1} \Xi_k X_{N-1}^{k,t,x} \]
\[ + \bar{T}_{k,N-1} \Xi_k X_{N-1}^{k,t,x} + \mu_{k,N-1} \]
Similarly, it holds that
\[ Z_{N-1} = \mathcal{R}_{N-1,N-1} + B_{N-1,N-1}^T \mathcal{G}_{N-1} + B_{N-1,N-1}, \]
\[ \mathcal{H}_{N-1} = B_{N-1,N-1}^T \mathcal{G}_{N-1} + B_{N-1,N-1}, \]
\[ \beta_{N-1} = B_{N-1,N-1}^T \mathcal{G}_{N-1} + B_{N-1,N-1} \]
\[ + \sum_{i,j=1}^p \gamma_{i,j}^N \mathcal{D}_{N-1,N-1} \mathcal{D}_{N-1,N-1} \]
\[ + \sum_{i,j=1}^p \gamma_{i,j}^N \mathcal{D}_{N-1,N-1} \mathcal{D}_{N-1,N-1} \]
\[ + \sum_{i,j=1}^p \gamma_{i,j}^N \mathcal{D}_{N-1,N-1} \mathcal{D}_{N-1,N-1}, \]
\[ \text{Note that } X_{N-1}^i \text{ is not influenced by } u_{N-1}^i, \]
From Lemma 3.4, \( u_{N-1}^i \) can be selected as
\[ u_{N-1}^i = -W_{N-1}^i \mathcal{H}_{N-1} X_{N-1}^{i,t,x} - W_{N-1}^i \beta_{N-1} \]
\[ \Delta \Psi_{N-1} X_{N-1}^{i,t,x} + \alpha_{N-1} \]
holds, which is equivalent to
\[ \mathcal{H}_{N-1} X_{N-1}^{i,t,x} + \beta_{N-1} = 0 \]
Therefore, we have
\[ \mathcal{E}_{N-2} Z_{N-1}^{N-2,t,x} \]
\[ = (P_{N-2,N-1} + T_{N-2,N-1}) X_{N-1}^{N-2,t,x} + \pi_{N-2,N-1} \]
\[ + \sum_{i,j=1}^p \gamma_{i,j}^N \mathcal{D}_{N-1,N-1} \mathcal{D}_{N-1,N-1} \]
\[ + \sum_{i,j=1}^p \gamma_{i,j}^N \mathcal{D}_{N-1,N-1} \mathcal{D}_{N-1,N-1} \]
\[ + \sum_{i,j=1}^p \gamma_{i,j}^N \mathcal{D}_{N-1,N-1} \mathcal{D}_{N-1,N-1}. \]
From (17), (63) and (64), we have
\[ 0 = \mathcal{W}_{N-1} u_{N-1}^{t,x} + \mathcal{H}_{N-1} X_{N-1}^{t,x} + \beta_{N-1}, \]
where

\[
\mathcal{W}_{N-2} = \mathcal{R}_{N-2,N-2} + \mathcal{B}^T_{N-2,N-2}\mathcal{P}_{N-2,N-1} + \mathcal{T}_{N-2,N-1}\mathcal{B}_{N-2,N-2} + \sum_{i,j=1}^p \gamma_{ij}^1 (\mathcal{D}^{ij}_{N-2,N-2})^T \times (P_{N-2,N-1} + T_{N-2,N-1}) \mathcal{D}^{ij}_{N-2,N-2},
\]

\[
\mathcal{H}_{N-2} = B^T_{N-2,N-2}(P_{N-2,N-1} + T_{N-2,N-1})A_{N-2,N-2} + \sum_{i,j=1}^p \gamma_{ij}^2 (\mathcal{D}^{ij}_{N-2,N-2})^T \times (P_{N-2,N-1} + T_{N-2,N-1}) \mathcal{D}^{ij}_{N-2,N-2},
\]

\[
\beta_{N-2} = B^T_{N-2,N-1}(P_{N-2,N-1} + T_{N-2,N-1})f_{N-2,N-2} + \pi_{N-2,N-1} + \rho_{N-2,N-2} + \sum_{i,j=1}^p \gamma_{ij}^2 (\mathcal{D}^{ij}_{N-2,N-2})^T (P_{N-2,N-1} + T_{N-2,N-1}) \times d_{ij}^{N-2,N-2}.
\]

From Lemma 3.4, \(u_{N-2}^{t,x,*}\) can be selected as

\[
u_{N-2}^{t,x,*} = -W_{N-2}^j \mathcal{H}_{N-2}X_{N-2}^{t,x,*} - W_{N-2}^j \beta_{N-2} \triangleq \Psi_{N-2}X_{N-2}^{t,x,*} + \alpha_{N-2}, \tag{70}\]

and

\[
(I - \mathcal{W}_{N-2})W_{N-2}^j \mathcal{H}_{N-2}X_{N-2}^{t,x,*} + \beta_{N-2} = 0,
\]

holds, which is equivalent to

\[
\mathcal{H}_{N-2}X_{N-2}^{t,x,*} + \beta_{N-2} \in \text{Ran}(\mathcal{W}_{N-2}).
\]

Backwardly repeating above procedure, we can get (29) and (34).

\[\text{ii)} \Rightarrow \text{i).}\] Let \(X_{N-2}^{t,x,*}\) and \(u_{N-2}^{t,x,*}\) be given in (33) (34). From (29), we have

\[
0 = \mathcal{W}_ku_{k,x}^{t,x,*} + \mathcal{H}_kX_{k,x}^{t,x,*} + \beta_{k,x}, \quad k \in \mathbb{T}_t.
\]

Furthermore, from (34) and Lemma 3.3, then, (26), equivalently (35), holds. Similarly to (63) and (64), we have

\[
\mathbb{E}_kZ_{k+1}^{k,t,x} = (\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1})\mathcal{A}_kX_{k,x}^{t,x,*} + (\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1})B_ku_{k,x}^{t,x,*} + (\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1})f_{k,x} + \pi_{k,k+1}, \tag{72}\]

and

\[
\mathbb{E}_k(Z_{k+1}^{k,t,x}u_k^{1,*}) = \sum_{j=1}^p \gamma_{kj}^1 (P_{k,k+1} + T_{k,k+1})C_{k,x}^{j,k}X_{k,x}^{t,x,*} + \sum_{j=1}^p \gamma_{kj}^1 (P_{k,k+1} + T_{k,k+1})D_{k,x}^{j,k}u_{k,x}^{t,x,*} + \sum_{j=1}^p \gamma_{kj}^1 (P_{k,k+1} + T_{k,k+1})d_{j,k}^{1,k}.
\]

Combining (71), (72) and (73), we have the stationary condition (17).